Sequences and Series

Arithmetic Sequences

Definitions

- Sequence: a list of numbers in a specific order.
- Term: each number in a sequence

Definitions

- Arithmetic Sequence: a sequence in which each term after the first term is found by adding a constant, called the common difference (d), to the previous term.

Explanations

- 150, 300, 450, 600, 750…
- The first term of our sequence is 150, we denote the first term as a_1.
- What is a_2?
- a_2 : 300 (a_2 represents the 2nd term in our sequence)

Explanations

- a_3 = ? a_4 = ? a_5 = ?
- a_3 : 450 a_4 : 600 a_5 : 750

- a_n represents a general term (nth term) where n can be any number.

Explanations

- Sequences can continue forever. We can calculate as many terms as we want as long as we know the common difference in the sequence.

Explanations

- Find the next three terms in the sequence:
  2, 5, 8, 11, 14, ___, ___, ___
- 2, 5, 8, 11, 14, **17**, **20**, **23**
- The common difference is?
- 3!!!

Explanations

- To find the common difference (d), just subtract any term from the term that follows it.
- FYI: Common differences can be negative.

Formula

- What if I wanted to find the 50th (a_{50}) term of the sequence 2, 5, 8, 11, 14, …? Do I really want to add 3 continually until I get there?
- There is a formula for finding the nth term.

Formula

- Let's see if we can figure the formula out on our own.
- $a_1 = 2$, to get a_2 I just add 3 once. To get a_3 I add 3 to a_1 twice. To get a_4 I add 3 to a_1 three times.

Formula

- What is the relationship between the term we are finding and the number of times I have to add d?
- The number of times I had to add is one less then the term I am looking for.

Formula

- So if I wanted to find a_{50} then how many times would I have to add 3?
- 49
- If I wanted to find a_{193} how many times would I add 3?
- 192

Formula

- So to find a_{50} I need to take d, which is 3, and add it to my a_1, which is 2, 49 times. That's a lot of adding.
- But if we think back to elementary school, repetitive adding is just multiplication.

Formula

- 3 + 3 + 3 + 3 + 3 = 15
- We added five terms of three, that is the same as multiplying 5 and 3.
- So to add three forty-nine times we just multiply 3 and 49.

Formula

- So back to our formula, to find a_{50} we start with 2 (a_1) and add 3•49. (3 is d and 49 is one less than the term we are looking for) So…
- $a_{50} = 2 + 3(49) = 149$

Formula

- $a_{50} = 2 + 3(49)$ using this formula we can create a general formula.
- a_{50} will become a_n so we can use it for any term.
- 2 is our a_1 and 3 is our d.

Formula

- $a_{50} = 2 + 3(49)$
- 49 is one less than the term we are looking for. So if I am using n as the term I am looking for, I multiply d by n - 1.

Formula

- Thus my formula for finding any term in an arithmetic sequence is $a_n = a_1 + d(n-1)$.
- All you need to know to find any term is the first term in the sequence (a_1) and the common difference.

Example

- Let's go back to our first example . Find therm number 16?

Example

- $a_n = a_1 + d(n-1)$
- We want to find a_{16}. What is a_1? What is d? What is n-1?
- a_1 = **150**, d = **150**,
  n -1 = 16 - 1 = **15**
- So a_{16} = 150 + 150(15) =
- $2400

Example

- 17, 10, 3, -4, -11, -18, …
- What is the common difference?
- Subtract any term from the term after it.
- -4 - 3 = -7
- d = - 7

Definition

- 17, 10, 3, -4, -11, -18, …
- Arithmetic Means: the terms between any two nonconsecutive terms of an arithmetic sequence.

Arithmetic Means

- 17, 10, 3, -4, -11, -18, …
- Between 10 and -18 there are three arithmetic means 3, -4, -11.
- Find three arithmetic means between 8 and 14.

Arithmetic Means

- So our sequence must look like 8, __, __, __, 14.
- In order to find the means we need to know the common difference. We can use our formula to find it.

Arithmetic Means

- 8, ___, ___, ___, 14
- $a_1 = 8$, $a_5 = 14$, & $n = 5$
- $14 = 8 + d(5 - 1)$
- $14 = 8 + d(4)$ subtract 8
- $6 = 4d$ divide by 4
- **$1.5 = d$**

Arithmetic Means

- 8, ___, ___, ___, 14 so to find our means we just add 1.5 starting with 8.
- 8, **9.5**, **11**, **12.5**, 14

Additional Example

- 72 is the ___ term of the sequence -5, 2, 9, …
- We need to find 'n' which is the term number.
- 72 is a_n, -5 is a_1, and 7 is d. Plug it in.

Additional Example

- $72 = -5 + 7(n - 1)$
- $72 = -5 + 7n - 7$
- $72 = -12 + 7n$
- $84 = 7n$
- $n = 12$
- 72 is the 12th term.

Arithmetic Series

Arithmetic Series

- The African-American celebration of Kwanzaa involves the lighting of candles every night for seven nights. The first night one candle is lit and blown out.

Arithmetic Series

- The second night a new candle and the candle from the first night are lit and blown out. The third night a new candle and the two candles from the second night are lit and blown out.

Arithmetic Series

- This process continues for the seven nights.
- We want to know the total number of lightings during the seven nights of celebration.

Arithmetic Series

- The first night one candle was lit, the 2nd night two candles were lit, the 3rd night 3 candles were lit, etc.
- So to find the total number of lightings we would add:
  1 + 2 + 3 + 4 + 5 + 6 + 7

Arithmetic Series

- 1 + 2 + 3 + 4 + 5 + 6 + 7 = 28
- **Series**: the sum of the terms in a sequence.
- **Arithmetic Series**: the sum of the terms in an arithmetic sequence.

Arithmetic Series

- Arithmetic sequence:
  2, 4, 6, 8, 10
- Corresponding arith. series:
  2 + 4 + 6 + 8 + 10
- Arith. Sequence: -8, -3, 2, 7
- Arith. Series: -8 + -3 + 2 + 7

Arithmetic Series

- S_n is the symbol used to represent the first 'n' terms of a series.
- Given the sequence 1, 11, 21, 31, 41, 51, 61, 71, … find S_4
- We add the first four terms 1 + 11 + 21 + 31 = 64

Arithmetic Series

- Find S_8 of the arithmetic sequence 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, …
- 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 =
- 36

Arithmetic Series

- What if we wanted to find S_{100} for the sequence in the last example. It would be a pain to have to list all the terms and try to add them up.
- Let's figure out a formula!! :)

Sum of Arithmetic Series

- Let's find S_7 of the sequence 1, 2, 3, 4, 5, 6, 7, 8, 9, …
- If we add S_7 in too different orders we get:

$$S_7 = 1 + 2 + 3 + 4 + 5 + 6 + 7$$

$$\underline{S_7 = 7 + 6 + 5 + 4 + 3 + 2 + 1}$$

$$2S_7 = 8 + 8 + 8 + 8 + 8 + 8 + 8$$

Sum of Arithmetic Series

$$S_7 = 1 + 2 + 3 + 4 + 5 + 6 + 7$$

$$S_7 = 7 + 6 + 5 + 4 + 3 + 2 + 1$$

$$2S_7 = 8 + 8 + 8 + 8 + 8 + 8 + 8$$

$$2S_7 = 7(8)$$ 7 sums of 8

$$S_7 = {}^{7}/_{2}(8)$$

Sum of Arithmetic Series

- $S_7 = {}^7/_2(8)$
- What do these numbers mean?
- 7 is n, 8 is the sum of the first and last term $(a_1 + a_n)$
- So $S_n = {}^n/_2(a_1 + a_n)$

Examples

- $S_n = {}^{n}/_{2}(a_1 + a_n)$
- Find the sum of the first 10 terms of the arithmetic series with $a_1 = 6$ and $a_{10} = 51$
- $S_{10} = 10/2(6 + 51) = 5(57) = 285$

Examples

- Find the sum of the first 50 terms of an arithmetic series with $a_1 = 28$ and $d = -4$
- We need to know n, a_1, and a_{50}.
- $n = 50$, $a_1 = 28$, $a_{50} = ??$ We have to find it.

Examples

- a_{50} = 28 + -4(50 - 1) = 28 + -4(49) = 28 + -196 = -168
- So n = 50, a_1 = 28, & a_n =-168
- S_{50} = (50/2)(28 + -168) = 25(-140) = -3500

Examples

- To write out a series and compute a sum can sometimes be very tedious. Mathematicians often use the greek letter sigma & summation notation to simplify this task.

Examples

last value of n

formula used to find sequence

$$\overset{5}{\underset{n=1}{!}}\; n + 1$$

First value of n

- This means to find the sum of the sums n + 1 where we plug in the values 1 - 5 for n

Examples

$$\sum_{n=1}^{5} n + 1$$

- Basically we want to find (1 + 1) + (2 + 1) + (3 + 1) + (4 + 1) + (5 + 1) =
- 2 + 3 + 4 + 5 + 6 =
- 20

Examples

- So $\mathop{!}\limits_{n=1}^{5} n+1=20$
- Try: $\mathop{!}\limits_{x=2}^{7} 3x - 2$
- First we need to plug in the numbers 2 - 7 for x.

Examples

$$\sum_{x=2}^{7} 3x - 2$$

- [3(2)-2]+[3(3)-2]+[3(4)-2]+ [3(5)-2]+[3(6)-2]+[3(7)-2] =
- (6-2)+(9-2)+(12-2)+(15-2)+ (18-2)+ (21-2) =
- 4 + 7 + 10 + 13 + 16 + 19 = 69

Geometric Sequences

GeometricSequence

- What if your pay check started at $100 a week and doubled every week. What would your salary be after four weeks?

GeometricSequence

- Starting $100.
- After one week - $200
- After two weeks - $400
- After three weeks - $800
- After four weeks - $1600.
- These values form a geometric sequence.

Geometric Sequence

- Geometric Sequence: a sequence in which each term after the first is found by multiplying the previous term by a constant value called the common ratio.

Geometric Sequence

- Find the first five terms of the geometric sequence with a_1 = -3 and common ratio (r) of 5.
- -3, -15, -75, -375, -1875

Geometric Sequence

- Find the common ratio of the sequence 2, -4, 8, -16, 32, …
- To find the common ratio, divide any term by the previous term.
- 8 ÷ -4 = -2
- r = -2

Geometric Sequence

- Just like arithmetic sequences, there is a formula for finding any given term in a geometric sequence. Let's figure it out using the pay check example.

Geometric Sequence

- To find the 5th term we look 100 and multiplied it by two four times.
- Repeated multiplication is represented using exponents.

Geometric Sequence

- Basically we will take \$100 and multiply it by 2^4
- $a_5 = 100 \cdot 2^4 = 1600$
- A_5 is the term we are looking for, 100 was our a_1, 2 is our common ratio, and 4 is n-1.

Examples

- Thus our formula for finding any term of a geometric sequence is $a_n = a_1 \cdot r^{n-1}$
- Find the 10th term of the geometric sequence with $a_1 = 2000$ and a common ratio of $^1/_2$.

Examples

- $a_{10} = 2000 \cdot (^1/_2)^9 =$
- $2000 \cdot {}^1/_{512} =$
- $^{2000}/_{512} = {}^{500}/_{128} = {}^{250}/_{64} = {}^{125}/_{32}$
- Find the next two terms in the sequence -64, -16, -4 ...

Examples

- -64, -16, -4, ___, ___
- We need to find the common ratio so we divide any term by the previous term.
- -16/-64 = 1/4
- So we multiply by 1/4 to find the next two terms.

Examples

- -64, -16, -4, -1, -1/4

Geometric Means

- Just like with arithmetic sequences, the missing terms between two nonconsecutive terms in a geometric sequence are called geometric means.

Geometric Means

- Looking at the geometric sequence 3, 12, 48, 192, 768 the geometric means between 3 and 768 are 12, 48, and 192.
- Find two geometric means between -5 and 625.

Geometric Means

- -5, ___, ___, 625
- We need to know the common ratio. Since we only know nonconsecutive terms we will have to use the formula and work backwards.

Geometric Means

- -5, ___, ___, 625
- 625 is a_4, -5 is a_1.
- $625 = -5 \cdot r^{4-1}$ divide by -5
- $-125 = r^3$ take the cube root of both sides
- $-5 = r$

Geometric Means

- -5, ___, ___, 625
- Now we just need to multiply by -5 to find the means.
- -5 • -5 = 25
- -5, 25, ___, 625
- 25 • -5 = -125
- -5, 25, -125, 625

Geometric Series

Geometric Series

- Geometric Series - the sum of the terms of a geometric sequence.
- Geo. Sequence: 1, 3, 9, 27, 81
- Geo. Series: 1+3 + 9 + 27 + 81
- What is the sum of the geometric series?

Geometric Series

- 1 + 3 + 9 + 27 + 81 = 121
- The formula for the sum S_n of the first n terms of a geometric series is given by

$$S_n = \frac{a_1 - a_1 r^n}{1 - r} \quad or \quad S_n = \frac{a_1(1 - r^n)}{1 - r}$$

Geometric Series

- Find $\sum_{n=1}^{4} -3(2)^{n-1}$
- You can actually do it two ways. Let's use the old way.
- Plug in the numbers 1 – 4 for n and add.
- $[-3(2)^{1-1}]+[-3(2)^{2-1}]+[-3(2)^{3-1}]+[-3(2)^{4-1}]$

Geometric Series

- [-3(1)] + [-3(2)] + [-3(4)] + [-3(8)] =
- -3 + -6 + -12 + -24 = -45
- The other method is to use the sum of geometric series formula.

Geometric Series

- $\sum_{n=1}^{4} -3(2)^{n-1}$ use $S_n = \dfrac{a_1(1-r^n)}{1-r}$

- $a_1 = -3$, $r = 2$, $n = 4$

Geometric Series

- $\overset{4}{\underset{n=1}{!}} -3(2)^{n-1}$ use $S_n = \dfrac{a_1(1-r^n)}{1-r}$
- $a_1 = -3,\ r = 2,\ n = 4$
-

$$S_4 = \frac{-3(1-2^4)}{1-2}$$

Geometric Series

- $$S_4 = \frac{-3(1-2^4)}{1-2}$$

$$S_4 = \frac{-3(1-16)}{-1}$$

$$S_4 = \frac{-3(-15)}{-1} = \frac{45}{-1} = -45$$